Code -

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(An Autonomous Institute)

END SEMESTER EXAMINATION

Second Year B.Tech (Branch) – Feb/Mar-2023

Course Code : BSC204 **Course Name : Linear Algebra and Transform**

Duration : 2 Hrs. **Max. Marks : 50** **Date :1/2/2023**

Instructions :

i) All questions are compulsory

ii) Assume suitable data wherever necessary and clearly state it

iii) Figures to right indicate full marks

iv) You can carry standard normal distribution table.

Q. 1	Answer any five(Marks:10)				
		Marks	CO	BL	PI
a)	Find the locus of z given by $\lvert z-3\rvert = 5$. Answer:Let $z = x + iy$ then $\lvert z-3\rvert = 5$ $\lvert x+iy-3\rvert = 5$ $\lvert x-3+iy\rvert = 5$ $\sqrt{(x-3)^2+y^2} = 5$ $(x-3)^2+y^2 = 5^2$ Comparing with $(x-a)^2+(y-b)^2 = k^2$, a circle with centre (a,b) and radius k we get locus is a circle with centre (3,0) and radius 5.	2	2	2	1.1.1, 2.1.3
b)	Express $z = 2+2i$ in polar form. Ans. $r = \sqrt{x^2+y^2} = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$ units. Amplitude $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \pi/4$. The polar form is $z = r[\cos\theta + i\sin\theta]$ $= 2\sqrt{2}[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}]$	2	2	2	1.1.1, 2.1.3
c)	Find the characteristic equation of the matrix A = $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$. Ans.: Characteristic equation of matrix A is $\lvert A-\lambda I\rvert = 0$. $\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = 0$ i.e. $(2-\lambda)(1-\lambda) - 4\times 3 = 0$ i.e. $\boxed{\lambda^2 - 3\lambda + 2 = 0}$.	2	3	2	1.1.1, 2.1.3, 12.1.1
d)	Write the formulae to find probability by	2	4	1	1.1.1, 2.1.3.

	i) Binomial distribution and ii) Poisson's distribution. Ans. i) *Probability of r times success in n trials by binomial distribution* $P(r) = n_{C_r} p^r q^{n-r}, r = 0,1,2,\ldots,n$ *where* $n_{C_r} = \frac{n!}{(n-r)! \times r!}$, *p is the probability of success in a single trial and* $q = 1 - p$. ii) *Probability of r times success by Poisson's distribution* $== P\,(r) = \frac{\lambda^r}{r!} e^{-\lambda}$ *where* $r = 0,1,2,\ldots,n$, $\lambda = np$, *n is number of trials, p is the probability of success in a single trial.*				12.1.1
e)	Find particular integral of $(D^2 + 2D + 1)y = 5$. Ans. Using the short method formula $\frac{1}{f(D)} k = \frac{1}{f(0)} \times k$ we get $P.I. = \frac{1}{(D^2+2D+1)} 5 = \frac{1}{(0+0+1)} 5 = 5$.	2	5	2	1.1.1, 2.1.3
f)	If $L\{f(t)\} = \frac{1}{s-1}$ then find $L\{f(3t)\}$. Ans. Using $L[f(at)] = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$ for $a = 3$ & $L\{f(t)\} = \frac{1}{s-1}$ we get $L\{f(3t)\} = \frac{1}{3}\left(\frac{1}{\frac{s}{3}-1}\right)$ $= \frac{1}{3}\left(\frac{3}{s-3}\right)$ $= \left(\frac{1}{s-3}\right)$	2	1	1	1.1.1, 2.1.3
g)	i) State Second shifting theorem of Laplace transform. If $L\{f(t)\} = \bar{f}(s)$ and $F(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$ then $L\{F(t)\} = e^{-as} \bar{f}(s)$. ii) Write the Laplace transform of Heaviside unit step function $H(t)$ and the Dirac delta function $\delta(t)$. $L\{H(t)\} = \frac{1}{s}$ and $L\{\delta(t)\} = 1$.	2	1	1	1.1.1, 2.1.3
h)	Find $L^{-1}\left\{\frac{s+1}{s^2-4}\right\}$. Ans. $L^{-1}\left\{\frac{s+1}{s^2-4}\right\} = L^{-1}\left\{\frac{s}{s^2-4}\right\} + L^{-1}\left\{\frac{1}{s^2-4}\right\}$ $= \cosh 2t + \frac{1}{2} \sinh 2t$.	2	6	2	1.1.1, 2.1.3

Q.2	a) If $\sin(\alpha + i\beta) = x + iy$ then prove that $\frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta} = 1.$	4	2	3	1.1.1, 2.1.3
	Ans. Given $\sin(\alpha + i\beta) = x + iy$. Using $\sin(A+B) = \sin A\cos B + \cos A\sin B$ on L.H.S we get $\sin\alpha\cos i\beta + \cos\alpha\sin i\beta = x + iy.$ Using $\sin i\theta = i\sinh\theta$ and $\cos i\theta = \cosh\theta$ we get	4			
	$\sin\alpha\cosh\beta + i\cos\alpha\sinh\beta = x + iy$...(1)	4			
	Equating real and imaginary parts on both sides of eq. (1) we get $\sin\alpha\cosh\beta = x$ and $\cos\alpha\sinh\beta = y$. Thus $\sin\alpha = \frac{x}{\cosh\beta}$ .... (2) and $\cos\alpha = \frac{y}{\sinh\beta}$ ....(3) Squaring and adding eq. (2) and eq. (3) and using $\sin^2\theta + \cos^2\theta = 1$ we get $sin^2\alpha + \cos^2\alpha = 1 = \frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta}.$ Hence proved.	4			
	b) Find all the values of $(1)^{\frac{1}{3}}$. Ans. Let $z = x + iy = 1$ then $x = 1, y = 0, r = \sqrt{x^2+y^2} = 1, \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{1}\right) = \tan^{-1}(0) = 0.$ The general polar form is $z = 1 = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$ $= 1[\cos(2n\pi + 0) + i\sin(2n\pi + 0)]$ $1 = [\cos(2n\pi) + i\sin(2n\pi)]$........ (1) Taking third root on both the sides of eq. (1) we get $1^{\frac{1}{3}} = [\cos(2n\pi) + i\sin(2n\pi)]^{\frac{1}{3}}$ Applying De-Moivre's Theorem we get $1^{\frac{1}{3}} = [\cos\left(\frac{2n\pi}{3}\right) + i\sin\left(\frac{2n\pi}{3}\right)]$ .... (2) Putting $n = 0,1,2$ in eq. (2) we get the three roots $R_1 = 1^{\frac{1}{3}} = [\cos(0) + i\sin(0)] = 1,$ $R_2 = 1^{\frac{1}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1}{2} + \frac{\sqrt{3}}{2}i,$ $R_3 = 1^{\frac{1}{3}} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = \frac{-1}{2} - \frac{\sqrt{3}}{2}i.$				
	OR(optional) a) Prove that $\log\frac{a+ib}{a-ib} = 2i\tan^{-1}\left(\frac{b}{a}\right).$				

$L.H.S = \log\dfrac{a+ib}{a-ib} = \log(a+ib) - \log(a-ib)$

Now using

$log(x+iy) = \frac{1}{2}\log(x^2+y^2) + \text{i}\ \tan^{-1}\left(\frac{y}{x}\right)$ we get

$log(a+ib) = \dfrac{1}{2}\log(a^2+b^2) + \text{i}\ \tan^{-1}\left(\dfrac{b}{a}\right)$

$log(a-ib) = \dfrac{1}{2}\log(a^2+b^2) - \text{i}\ \tan^{-1}\left(\dfrac{b}{a}\right)$

$\therefore\ L.H.S = \left[\dfrac{1}{2}\log(a^2+b^2) + \text{i}\ \tan^{-1}\left(\dfrac{b}{a}\right)\right]$

$\qquad - \left[\dfrac{1}{2}\log(a^2+b^2) - \text{i}\ \tan^{-1}\left(\dfrac{b}{a}\right)\right]$

$= 2\text{i}\ \tan^{-1}\left(\frac{b}{a}\right)$ Hence proved.

b) Prove that $\cosh^{-1}x = \log(x+\sqrt{x^2-1})$.

Ans. Let $\cosh^{-1}x = y$.

$\therefore x = \cosh y = \dfrac{e^y + e^{-y}}{2}$

$\therefore 2x = e^y + \dfrac{1}{e^y} = \dfrac{e^{2y}+1}{e^y}$

$\therefore 2xe^y = e^{2y} + 1$

$\therefore e^{2y} - 2xe^y + 1 = 0$

Put $e^y = t \therefore t^2 - 2xt + 1 = 0$

$\therefore t = \dfrac{2x \pm \sqrt{4x^2-4}}{2}$

$= \dfrac{2x \pm 2\sqrt{x^2-1}}{2}$

$t = x \pm \sqrt{x^2-1}$

Taking positive sign only

$t = x + \sqrt{x^2-1},$

$\therefore e^y = x + \sqrt{x^2-1}$

$\therefore \log e^y = \log\left(x+\sqrt{x^2-1}\right)$

$\therefore y\log e = \log\left(x+\sqrt{x^2-1}\right)$

$\therefore y = \log\left(x+\sqrt{x^2-1}\right)$

But $y = \cosh^{-1}x$

$\therefore \cosh^{-1}x = \log\left(x+\sqrt{x^2-1}\right).$

Hence proved.

Q.3	a) If the characteristic equation of a matrix$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ is $A^3 - 6A^2 + 7A + 2I = 0$ then verify Cayley-Hamilton	4	3	3	1.1.1, 2.1.3, 12.1.1

theorem for A.

Ans. $A^2 = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$, 4

$\therefore 6A^2 = \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix}$

$A^3 = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$,

$7A = 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix}$ 8

$2I = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$L.H.S A^3 - 6A^2 + 7A + 2I = 0 = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} -$

$\begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} =$

$\begin{bmatrix} 21-30+7+2 & 0-0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0-0+0+0 & 55-78+21+2 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence verified.

b) Solve the given simultaneous equations :

$2x - y + 3z = 0,$

$3x + 2y + z = 0, x - 4y + 5z = 0.$

Ans. The given system can be written in matrix form as:

$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

where $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and the

augmented matrix $C = [A|B]$

$C = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & -4 & 5 & 0 \end{array}\right].$

$R_1 \leftrightarrow R_3$

$\sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$

$R_2 - 3R_1, R_3 - 2R_1$

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$$\sim\left[\begin{array}{ccc|c}1 & -4 & 5 & 0\\ 0 & 14 & -14 & 0\\ 0 & 7 & -7 & 0\end{array}\right]$$

$\frac{R_2}{14}, \frac{R_3}{7}$

$$\sim\left[\begin{array}{ccc|c}1 & -4 & 5 & 0\\ 0 & 1 & -1 & 0\\ 0 & 1 & -1 & 0\end{array}\right]$$

$R_3 - R_2$

$$\sim\left[\begin{array}{ccc|c}1 & -4 & 5 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

This is echelon form. $\rho(A) = \rho(C) = 2 <$ number of unknowns $n = 3$. Thus, the given homogeneous system is consistent and has infinitely many solutions. Forming equations from echelon form we get

$x - 4y + 5z = 0 \ldots\ldots (1)$

$y - z = 0 \ldots\ldots (2)$

Let $z = k \ldots\ldots (3)$ where k is any constant.

Using eq.(3) in eq.(2) we get $y - k = 0$

$\therefore y = k \ldots\ldots (4)$

Using eq.(4) and eq.(3) in eq. (1) we get

$x - 4k + 5k = 0$

$\therefore x + k = 0$

$\therefore x = -k.$

Thus infinite solutions are $\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-k\\ k\\ k\end{bmatrix}$.

OR(optional)

Find the eigen values and eigen vector corresponding to the smallest eigen value of the matrix $A = \begin{bmatrix}2 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}$.

Ans. Characteristic equation of the matrix is $|A - \lambda I| = 0$.

$$\begin{vmatrix}2-\lambda & 0 & 1\\ 0 & 2-\lambda & 0\\ 1 & 0 & 2-\lambda\end{vmatrix} = 0.$$

$(2-\lambda)\{(2-\lambda)(2-\lambda)\} + 1\{-(2-\lambda)\} = 0$

$(2-\lambda)\{(2)(2-\lambda) - \lambda(2-\lambda)\} + 1\{-(2-\lambda)\} = 0$

$(2-\lambda)\{4 - 2\lambda - 2\lambda + \lambda^2\} + \{\lambda - 2\} = 0$

$(2-\lambda)\{4 - 4\lambda + \lambda^2\} + \{\lambda - 2\} = 0$

	$(2)\{4-4\lambda+\lambda^2\}-\lambda\{4-4\lambda+\lambda^2\}+\{\lambda-2\}=0$ $\{8-8\lambda+2\lambda^2]-\{4\lambda-4\lambda^2+\lambda^3\}+\{\lambda-2\}=0$ $8-8\lambda+2\lambda^2-4\lambda+4\lambda^2-\lambda^3\}+\lambda-2=0$ $8-8\lambda+2\lambda^2-4\lambda+4\lambda^2-\lambda^3+\lambda-2=0$ $8-8\lambda+2\lambda^2-4\lambda+4\lambda^2-\lambda^3+\lambda-2=0$ $-\lambda^3+6\lambda^2-11\lambda+6=0$ $\lambda^3-6\lambda^2+11\lambda-6=0.$ Solving this for λwe get the eigen values as $\lambda=1,2,3$. Eigen vector corresponding to $\lambda=1$ is given by $[A-\lambda I]X=0$ for $\lambda=1$i.e. $\begin{bmatrix} 2-1 & 0 & 1 \\ 0 & 2-1 & 0 \\ 1 & 0 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $x_1+x_3=0$ $x_2=0$ $x_1+x_3=0$ Let $x_1=k$ then $k+x_3=0$ gives $k+x_3=0$ $x_3=-k$ $x_2=0$ Thus eigen vector for $\lambda=1$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} k \\ 0 \\ -k \end{bmatrix}$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=$ $\begin{bmatrix} -k \\ 0 \\ k \end{bmatrix}$				
Q.4	a) Solve $\frac{d^2y}{dx^2}-4y=x\sinh x$.	4	5	3	4
	Solution: Converting given equation in symbolic form we get $(D^2-4)y=x\sinh x$. General solution $y=C.F.+P.I.$	4			4
	C.F.:$(D^2-4)=0$. The roots of this equation are $D=2,-2$. Two real and different roots. Thus, $C.F.=(c_1e^{2x}+c_2e^{-2x})$ P.I.: $P.I.=\frac{1}{(D^2-4)}x\sinh x$	8			8

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We use the formula $\frac{1}{f(D)}xV = \left[x - \frac{f'(D)}{f(D)}\right]\left[\frac{1}{f(D)}\right]V$.

Here $V = \sinh x, f(D) = D^2 - 4, f'(D) = 2D$

$$\therefore P.I. = \frac{1}{(D^2-4)} x \sinh x$$

$$= \left[x - \frac{2D}{D^2-4}\right]\left\{\left[\frac{1}{D^2-4}\right]\sinh x\right\}$$

Using the formula $\frac{1}{f(D^2)}\sinh(ax+b) = \frac{1}{f(a^2)} \times$ $\sinh(ax+b)$ provided $f(a^2) \neq 0$ with $a = 1$.

$$\therefore P.I. = \left[x - \frac{2D}{D^2-4}\right]\left\{\left[\frac{1}{1^2-4}\right] \times \sinh x\right\}$$

$$\therefore P.I. = \left[x - \frac{2D}{D^2-4}\right]\left\{\left[\frac{-1}{3}\right] \times \sinh x\right\}$$

$$= \frac{-1}{3}\left[x - \frac{2D}{D^2-4}\right]\sinh x$$

$$= \frac{-1}{3}\left[x \times \sinh x - 2D\frac{1}{D^2-4}\sinh x\right]$$

(Again, applying the formula $\frac{1}{f(D^2)}\sinh(ax+b) = \frac{1}{f(a^2)} \times$ $\sinh(ax+b)$ we get)

$$P.I. = \frac{-1}{3}\left[x \times \sinh x - 2D\frac{1}{D^2-4}\sinh x\right]$$

$$P.I. = \frac{-1}{3}\left[x \sinh x - 2D\frac{1}{1^2-4}\sinh x\right]$$

$$= \frac{-1}{3}\left[x \sinh x - 2D\frac{1}{-3}\sinh x\right]$$

$$= \frac{-1}{3}\left[x \sinh x + \frac{2}{3}D \sinh x\right]$$

$$= \frac{-1}{3}\left[x \sinh x + \frac{2}{3}D \sinh x\right]$$

$$= \frac{-1}{3}\left[x \sinh x + \frac{2}{3}\cosh x\right]$$

$$= \frac{-1}{9}[3x \sinh x + 2\cosh x]$$

$$= \frac{-[3x \sinh x + 2\cosh x]}{9}.$$

Thus, general solution

$$y = (c_1 e^{2x} + c_2 e^{-2x}) - \frac{[3x \sinh x + 2\cosh x]}{9}.$$

b) Solve$(D^2 + 3D + 2)y = e^{e^x}$.

Solution: C.F. is as in above example $C.F. = c_1 e^{-2x} + c_2 e^{-x}$.

Comparing with $C.F. = c_1 f_1 + c_2 f_2$ we get

$f_1 = e^{-2x}, f_2 = e^{-x}$

$f_1' = \frac{d}{dx}e^{-2x} = -2\,e^{-2x}$ and $f_2' = \frac{d}{dx}e^{-x} = -e^{-x}$

$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = f_1 f_2' - f_1' f_2.$

$W(f_1, f_2) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}$

$= e^{-2x}(-e^{-x}) - (-2e^{-2x}e^{-x})$

$= -e^{-2x-x} + 2e^{-2x-x}$

$= -e^{-3x} + 2e^{-3x}$

$= e^{-3x}.$

$u = -\int \frac{f_2 X}{W(f_1, f_2)} dx$

$u = -\int \frac{e^{-x}e^{e^x}}{e^{-3x}} dx$

$u = -\int e^{3x-x}e^{e^x}\,dx$

$= -\int e^{2x}e^{e^x}\,dx$

$= -\int e^x e^{e^x} e^x\,dx.$

Let $e^x = t$ then $e^x dx = dt$. Using this we get,

$u = -\int te^t dt.$

Here t is algebraic function - A, e^t is exponential function-E.

According to LIATE rule $u = t, v = e^t$.

Note that this notation has nothing to do with u of our M.O.V.O.P

We use the formula product rule of integration

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \ldots\ldots$

where $v_1 = \int v,\ v_2 = \int v_1$ and so on.

$u = -[t\,e^t - 1e^t]$

$u = -[t\ - 1]e^t$

$u = [1 - t]e^t$

$u = [1 - e^x]e^{e^x}$

$v = \int \frac{f_1 X}{W(f_1, f_2)} dx$

$v = \int \frac{e^{-2x}e^{e^x}}{e^{-3x}} dx$

$= \int e^{3x-2x}e^{e^x} dx$

$= \int e^x e^{e^x} dx$

$= \int e^{e^x} e^x dx$

Let $e^x = t$ then $e^x\,dx = dt$. Thus,

$v = \int e^t dt$

$v = e^t$

$v = e^{e^x}$.

By M.O.V.O.P, $P.I. = uf_1 + vf_2$

$\therefore P.I. = [1 - e^x]e^{e^x}e^{-2x} + e^{e^x}e^{-x}$

i.e. $P.I. = e^{e^x}e^{-2x} - e^x e^{e^x}e^{-2x} + e^{e^x}e^{-x}$

i.e.$P.I. = e^{e^x}e^{-2x} - e^{-x}e^{e^x} + e^{e^x}e^{-x}$

i.e. $\therefore P.I. = e^{e^x}e^{-2x}$

Thus, the general solution

$y = C.F. + P.I. = c_1e^{-2x} + c_2e^{-x} + e^{e^x}e^{-2x}$

OR(optional)

An electric circuit consists of an inductance L, a condenser of capacitance C and e.m.f $E = E_0 \cos \omega t$ so that the charge Q saisfies the differential equation $\frac{d^2Q}{dt^2} + \frac{Q}{LC} = \frac{E_0}{L} \cos \omega t$ if $\omega = \frac{1}{\sqrt{LC}}$ and initially at $t = 0, Q = Q_0$ and current $i = i_0$ find the charge at any time t.

Ans. $\frac{d^2Q}{dt^2} + \frac{Q}{LC} = \frac{E_0}{L} \cos \omega t$

$$\frac{d^2Q}{dt^2} + \omega^2 Q = \frac{E_0}{L} \cos \omega t$$

$$(D^2 + \omega^2)Q = \frac{E_0}{L} \cos \omega t$$

$$(D^2 + \omega^2)Q = \frac{E_0}{L} \cos \omega t$$

A.E. is $(D^2 + \omega^2) = 0$

$D^2 = -\omega^2$

$D = \pm \omega i$

$C.F. = c_1 \cos \omega t + c_2 \sin \omega t$

$$P.I. = \frac{1}{(D^2 + \omega^2)} \frac{E_0}{L} \cos \omega t$$

$$= \frac{E_0}{L} \frac{1}{(D^2 + \omega^2)} \cos \omega t$$

$$= \frac{E_0}{L} \frac{1}{(-\omega^2 + \omega^2)} \cos \omega t$$

$$= \frac{E_0}{L} \frac{1}{(0)} \cos \omega t$$

$$= \frac{E_0 t}{L} \frac{1}{(2D)} \cos \omega t$$

$$= \frac{E_0 t}{2L} \frac{1}{D} \cos \omega t$$

$$= \frac{E_0 t}{2L} \frac{\sin \omega t}{\omega}$$

$$= \frac{E_0 t \sin \omega t}{2L\omega}.$$

	General solution is $Q = c_1\cos\omega t + c_2\sin\omega t + \frac{E_0 t}{2L}\frac{\sin\omega t}{\omega}$. $i = -c_1\omega\sin\omega t + c_2\omega\cos\omega t + \frac{E_0}{2L\omega}[t\omega\cos\omega t + \sin\omega t]$ Using $Q = Q_0$ when $t = 0$ we get $c_1 = Q_0$. Using $i = i_0$ when $t = 0$ we get $i_0 = c_2\omega$ i.e. $c_2 = \frac{i_0}{\omega}$. Thus charge is $Q = Q_0\cos\omega t + \frac{i_0}{\omega}\sin\omega t + \frac{E_0 t\sin\omega t}{2L\omega}$				
Q.5	a) Evaluate the integral $\int_0^\infty e^{-3t}\sin t\,dt$ Solution: Let $I = \int_0^\infty e^{-3t}\sin t\,dt$. On comparing with $L[f(t)] = \int_0^\infty e^{-st}f(t)dt$ we get $s = 3$ and $f(t) = \sin t$ So that, $I = \int_0^\infty e^{-3t}\sin t\,dt = L\{\sin t\}_{s=3}$ $\because L\{\sin at\} = \frac{a}{s^2+a^2}$ with $a = 1$ we get , $I = \int_0^\infty e^{-3t}\sin t\,dt = L\{\sin t\}_{s=3} = \left(\frac{1}{s^2+1}\right)_{s=3} =$ $\left(\frac{1}{3^2+1}\right) = I = \left(\frac{1}{10}\right)$. b) Find $L^{-1}\left[\frac{s+3}{(s+4)^2}\right]$. $L^{-1}\left[\frac{s+3}{(s+4)^2}\right] = L^{-1}\left[\frac{(s+4)-1}{(s+4)^2}\right]$ $\because\ L^{-1}[f(s+a)] = e^{-at}L^{-1}[F(s)]$ $\therefore L^{-1}\left[\frac{s+3}{(s+4)^2}\right] = e^{-4t}L^{-1}\left[\frac{s-1}{s^2}\right]$ $= e^{-4t}L^{-1}\left[\frac{1}{s} - \frac{1}{s^2}\right]$ $\because L^{-1}\left[\frac{1}{s}\right] = 1$ and $e^{-4t}L^{-1}\left[\frac{1}{s^2}\right] = t$ $\therefore L^{-1}\left[\frac{s+3}{(s+4)^2}\right] = e^{-4t}[1-t]$. **OR**(optional) a)Find $L[\sin^2 4t]$. Ans. $L[\sin^2 4t] = L\left[\frac{1-\cos 8t}{2}\right]$ $= \frac{1}{2}L[1-\cos 8t]$ $= \frac{1}{2}\{L[1] - L[\cos 8t]\} \because L[1] = \frac{1}{s}$ *and* $L[\cos at] = \frac{s}{s^2+a^2}$	4 4 4 4	5	3	1.1.1, 2.1.3

	$\therefore L[\sin^2 4t] = \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+8^2}\right\}$ $= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+64}\right\}$ $= \frac{1}{2}\left\{\frac{s^2+64-s^2}{s(s^2+64)}\right\}$ $= \frac{1}{2}\left\{\frac{64}{s(s^2+64)}\right\}$ $= \frac{32}{s(s^2+64)}.$ b) Obtain inverse Laplace transform of $\frac{a}{s(s-a)}$. Ans. $L^{-1}\left[\frac{a}{s(s-a)}\right] = L^{-1}\left[\frac{1}{s}\frac{a}{(s-a)}\right]$. Comparing with $L^{-1}[\bar{f}_1(s)\bar{f}_2(s)] = \int_0^t f_1(t-u)f_2(u)\,du$ we get $\bar{f}_1(s) = \frac{1}{s}, f_1(t-u) = L^{-1}\left[\frac{1}{s}\right] = 1$ $\bar{f}_2(s) = \frac{a}{(s-a)}, f_2(u) = L^{-1}\left[\frac{a}{s-a}\right] = aL^{-1}\left[\frac{1}{s-a}\right] = ae^{at}$ $\therefore L^{-1}\left[\frac{a}{s(s-a)}\right] = \int_0^t 1ae^{at}\,du$ $= \int_0^t ae^{at}\,du$ $= a\int_0^t e^{at}\,du$ $= \frac{a}{a}[e^{at}-1]$ $= e^{at} - 1.$				

Q.6	a) Six dice are thrown 780 times. How many times do you expect at least four dice to show a three or four?	8				

Ans. Given $n = 6$

$N = 729$

at least 4 means 4 or 5 or 6

Firstly,

Probability of getting 3 or 4 on a single die

$= p = probability\ of\ \ getting\ 5 + probability\ of\ 6$

$= \frac{1}{6} + \frac{1}{6}$

$= \frac{2}{6}$

$p = \frac{1}{3}$

$q = 1 - \frac{1}{3} = \frac{2}{3}$

$expected\ number\ of\ times\ at\ least\ four\ dice\ showing$ $or\ 4 = N \times \boldsymbol{P(at\ least\ four\ dice\ showing\ 3\ or\ 4)}$

$= N \times [P(4) + P(5) + P(6)]$

Where

$P(4) = 6_{C_4} p^4 q^{6-4} = 6_{C_4} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 = 0.082305$

$P(5) = 6_{C_5} p^5 q^{6-5} = 6_{C_4} \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^1 = 0.016461$

$P(6) = 6_{C_6} p^6 q^{6-6} = 6_{C_4} \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^0 = 0.001372$

$\therefore expected\ number\ of\ times\ at\ least\ four\ dice$

$showing\ 3\ or\ 4$

$= N \times \boldsymbol{P(at\ least\ four\ dice\ showing\ 3\ or\ 4)}$

$= 780 \times [0.248287]$

$= 193.66 \sim 194$

b) Assuming that the diameter of 1000 brass plugs taken consecutively from a machine form a normal distribution with mean 0.7515 inch and S.D. 0.0020 inch, how many plugs are likely to be rejected if approved diameters are $0.7520\ \pm 0.0040$. Given that the area under the normal curve for $S.N.V. 1.75$ is 0.4599 and for $S.N.V. 2.25$ is 0.4878.

Ans. Given $\mu = 0.7515,\ \sigma = 0.0020$,

Approved diameters are $0.7520\ \pm 0.0040$ means diameters in between $(0.7520 - 0.0040)$ and $(0.7520 +$

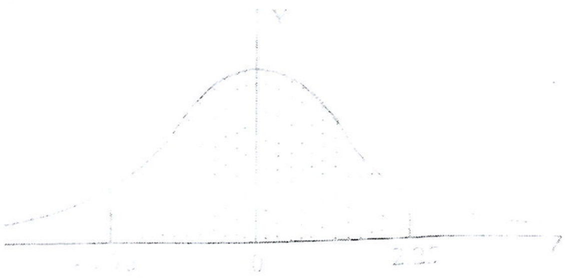
0.0040) are acceptable. Diameters other than in this range will be rejected.

$$x_1 = 0.7520 + 0.0040 = 0.7480, z_1 = \frac{0.7480 - 0.7515}{0.0020}$$

$$= -1.75$$

$$x_2 = 0.7520 - 0.0040 = 0.7560, z_2 = \frac{0.7560 - 0.7515}{0.0020}$$

$$= 2.25$$



Probability of plugs with approved diameter $=$ $P(0.7480 \leq x \leq 0.7560)$

$= Area\ (from\ z = 0\ up\ to\ z = -1.75) + Area\ (from\ z = 0\ to\ z = 2.25)$

$= Area\ (from\ z = 0\ to\ z = 1.75) + +Area\ (from\ z = 0\ to\ z = 2.25)$

$= 0.4599 + 0.4878$

$= 0.9477$

∴Number of brass plugs with approved diameters $=$ $1000 \times P(0.7480 \leq x \leq 0.7560)$

$= 1000 \times 0.9947$

$= 947.7 \approx 948$

Number of brass plugs that will be refused $= 1000 -$ Number of brass plugs that will be approved $= 1000 - 948 = 52$

OR(optional)

a) The probability that a man aged 35 years will die before reaching the age of 40 years may be taken as 0.018. Out of 400 men now aged 35 years, what is the probability that **2** men will die within the next 5 years?

Ans. Here $p = 0.018$, $n = 400$, $parameter\ \lambda = np = 400 \times 0.018 = 7.2$

Poisson's distribution formula is

$$P(r) = \frac{\lambda^r}{r!} e^{-\lambda}, r = 0,1,2, \ldots$$

Here formula of Poisson's distribution for death before the age of 40

$= P(r) = \frac{7.2^r}{r!} e^{-7.2}, r = 0,1,2, \dots, n$

$r = 2$ ∴Probability that 2 men will die

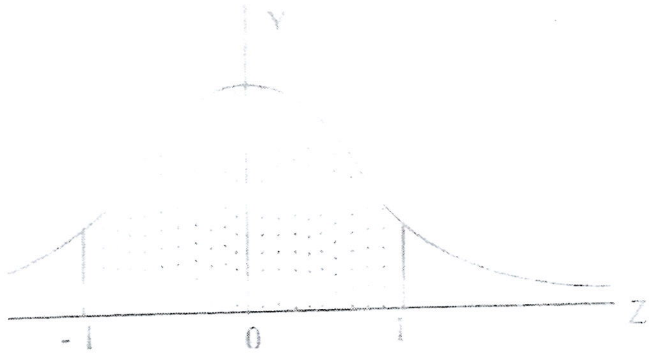
$= P(2) = \frac{(7.2)^2}{2!} e^{-7.2} = 0.01936.$

b) Normal distribution has mean 15.73 and S.D. 2.08. Find the percentage of cases that fall between 17.81 and 13.65. Given that the area under the normal curve for $S.N.V.\ 1$ is 0.3413.

Ans. Given $\mu = 15.73,\ \sigma = 2.08,$

$x_1 = 17.81, \quad z_1 = \frac{17.81 - 15.73}{2.08} = 1$

$x_2 = 13.65, \quad z_2 = \frac{13.65 - 15.73}{2.08} = -1$



Probability of cases that fall in between 17.81 and 13.65

$= P(-1 \le x \le 1)$

$= area\ between\ z_1 = 1\ and\ z_2 = -1$

$= Area\ (from\ z = 0\ up\ to\ z_1 = 1) + Area\ (from\ z = 0\ to\ z_2 = -1)$

$= Area\ (from\ z = 0\ up\ to\ z_1 = 1) + Area\ (from\ z = 0\ to\ z_2 = 1)$

$= 2 \times Area\ (from\ z = 0\ up\ to\ z_1 = 1)$

$= 2 \times 0.3413$

$= 0.6826$

∴ Required percentage $= 0.6826 \times 100 = 68.26\%$.